# Parity Path Signatures in Iterated HD Systems

Consider the normalized iterated system

$$x_{t+1} = \hat{H} D x_t$$

with

$$\hat{H} = \frac{1}{\sqrt{N}} H$$

where H is the Walsh-Hadamard matrix and D is a diagonal ±1 sign matrix.

In the standard path interpretation, each output component after k iterations is a sum over exponentially many computational paths through the recursive Walsh-Hadamard interaction graph.

For a unit input contribution, each individual path contributes a value of the form

$$\prod_{t=1}^{k} \sigma_t \, N^{-k/2}$$

where:

- each $\sigma_t$ $in$ $+1, -1,$
- and the sign sequence is determined by the parity structure encountered along the path through successive $H$ and $D$ operations.

The important point is that:

- the path sign history is not lost,
- it becomes encoded into the final arithmetic value through coherent signed accumulation.

Each iteration contributes:

1. a new parity-dependent sign selection,
2. followed by another normalization scaling factor.

Thus:

- deeper parity interactions contribute progressively smaller magnitude terms,
- while the number of such terms grows exponentially.

The repeated normalization by

$$\frac{1}{\sqrt{N}}$$

acts somewhat like repeated right shifting of arithmetic significance:

- early path decisions affect high-significance structure,
- deeper path decisions affect progressively lower-significance structure.

From this viewpoint, the final output value may be interpreted as containing an embedded multiscale record of parity interaction history.

In particular:

- the sequence of encountered (±1) parity interactions along a path becomes imprinted into the fine-scale arithmetic structure of the final coefficient,
- with deeper path structure appearing at progressively finer significance levels.

For Walsh-Hadamard systems this is natural because the transform itself is built from binary parity functions:

$$H_{a,b} = (-1)^{a \cdot b}$$

so the iterated dynamics repeatedly remix XOR/parity structure across increasingly deep interaction scales.

The unit-input case makes the mechanism easiest to visualize because the final coefficient structure then reflects only:

- parity pathway structure,
- sign accumulation,
- and normalization depth.

More generally, a non-unit input simply scales the contribution carried by each path while preserving the same parity-signature mechanism.

Thus iterated (HD) systems may be viewed as recursively encoding parity pathway histories into progressively finer arithmetic structure of the output state.